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# THE BRAUER GROUP OF GRADED CONTINUOUS TRACE $C^*$-ALGEBRAS

ELLEN MAYCOCK PARKER

ABSTRACT. Let $X$ be a locally compact Hausdorff space. The graded Morita equivalence classes of separable, $\mathbf{Z}_2$-graded, continuous trace $C^*$-algebras which have spectrum $X$ form a group, $\mathrm{GBr}^\infty(X)$, the infinite-dimensional graded Brauer group of $X$. Techniques from algebraic topology are used to prove that $\mathrm{GBr}^\infty(X)$ is isomorphic via an isomorphism $w$ to the direct sum $\check{H}^1(X;\underline{\mathbf{Z}_2}) \oplus \check{H}^3(X;\underline{\mathbf{Z}})$. The group $\mathrm{GBr}^\infty(X)$ includes as a subgroup the ungraded continuous trace $C^*$-algebras, and the Dixmier-Douady invariant of such an ungraded $C^*$-algebra is its image in $\check{H}^3(X;\underline{\mathbf{Z}})$ under $w$.

**Introduction.** The study of graded $C^*$-algebras has become particularly important since G. G. Kasparov's development of $KK$-theory for operator algebras [**18**]. In this paper, separable, $\mathbf{Z}_2$-graded, continuous trace $C^*$-algebras are classified. The graded Morita equivalence classes of such algebras whose spectra are all the same locally compact Hausdorff space $X$ form a group, called the infinite-dimensional graded Brauer group of $X$ and denoted by $\mathrm{GBr}^\infty(X)$. Two invariants defined on $\mathrm{GBr}^\infty(X)$ provide useful insights into the structure of these $C^*$-algebras and relate the results presented here to previous work.

The constructions of J. Dixmier and A. Douady [**3**, **4**, **5**] form an important framework for the graded classification. Let $X$ be a locally compact Hausdorff space, with countable base. Dixmier and Douady considered separable, stable, continuous trace $C^*$-algebras, with spectrum $X$. There is a canonical way to associate such an algebra $A$ with a fiber bundle $\xi_A$ over $X$ with fiber $\mathscr{K}(\mathscr{H})$, the compact operators on an infinite-dimensional separable Hilbert space. Let $\mathscr{P}\mathscr{U}(\mathscr{H})$ be the projective unitary group of $\mathscr{H}$, and let $\check{H}^*(X;\underline{G})$ denote the Čech cohomology of $X$ with coefficients in the sheaf of germs of continuous functions from $X$ to $G$, for $G$ a group. Then the isomorphism class of $\xi_A$ is an element of $\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}(\mathscr{H})})$, which can be shown to be isomorphic to $\check{H}^3(X;\underline{\mathbf{Z}})$. They defined the Dixmier-Douady invariant $\delta(A) \in \check{H}^3(X;\underline{\mathbf{Z}})$ of the algebra $A$, and proved that the invariant defines a one-to-one correspondence between isomorphism classes of such algebras and the elements of $\check{H}^3(X;\underline{\mathbf{Z}})$.

Consider now the collection of graded, separable, continuous trace $C^*$-algebras, all with spectrum $X$. We will define $\mathrm{GBr}^\infty(X)$ as the set of equivalence classes of all such $C^*$-algebras under graded Morita equivalence, which is the graded version of strong Morita equivalence defined by M. Rieffel [**22**, **23**]. It is important to note,

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however, that each equivalence class of $\mathrm{GBr}^{\infty}(X)$ can be uniquely represented, up to spectrum-preserving graded *-isomorphism, by a $C^*$-algebra which is a separable, graded, stable, continuous trace $C^*$-algebra, with spectrum $X$. In the first sections of the paper, we choose such a representation, and delay a more thorough discussion of graded Morita equivalence until §5.

Let $A$ be a separable, graded, stable, continuous trace $C^*$-algebra, with spectrum $X$. In this paper, a graded fiber bundle $\xi_A$ is constructed from $A$ using techniques parallel to those in the ungraded case. If $x \in X$ is an irreducible representation, then $A/\ker(x)$ is shown to be isomorphic, via a map which preserves the grading, to $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$, the graded compact operators on a separable, infinite-dimensional graded Hilbert space $\mathscr{H}$. The fiber of $\xi_A$ over $x$ is then $A/\ker(x)$. A topology on the total space $E(\xi_A)$ is given, and a structure group $\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ for the bundle is defined. The original algebra $A$ can be retrieved by considering the set of sections of $\xi_A$ which vanish at $\infty$. The correspondence between $A$ and $\xi_A$ lies at the heart of the main result: that $\mathrm{GBr}^{\infty}(X)$ is isomorphic to $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$, which in turn is isomorphic to the direct sum $\check{H}^1(X; \underline{\mathbf{Z}}_2) \oplus \check{H}^3(X; \underline{\mathbf{Z}})$. The isomorphism

$$w\colon\ \mathrm{GBr}^{\infty}(X) \to \check{H}^2(X; \underline{\mathbf{Z}}_2) \oplus \check{H}^3(X; \underline{\mathbf{Z}})$$

defines invariants $w_1^*(A) \in \check{H}^1(X; \underline{\mathbf{Z}}_2)$ and $w_2^*(A) \in \check{H}^3(X; \underline{\mathbf{Z}})$ for $A \in \mathrm{GBr}^{\infty}(X)$. When $A$ is ungraded, it is shown that $w_2^*(A) = \delta(A)$ and $w_1^*(A) = 1$. The group structure is analyzed and an explicit inverse to an element of $\mathrm{GBr}^{\infty}(X)$ is constructed.

The correspondence between graded continuous trace $C^*$-algebras and graded fiber bundles allows the finite-dimensional cases considered by J.-P. Serre [**12**], P. Donovan and M. Karoubi [**6**], and R. Patterson [**19**, **20**] to be included in $\mathrm{GBr}^{\infty}(X)$. The invariants of Donovan and Karoubi agree with those defined here. In addition, by applying the work of J. Phillips and I. Raeburn [**21**], it is shown that $w_1^*(A)$ is the obstruction to the grading automorphism of $A$ being an inner automorphism. Using a construction of P. Green [**11**] for the correspondence between the isomorphism classes of continuous trace $C^*$-algebras and $\check{H}^3(X; \underline{\mathbf{Z}})$, an alternate definition for the isomorphism $w$ is given. This definition allows some modifications in the equivalence relation on $\mathrm{GBr}^{\infty}(X)$ to be made. Further applications of the infinite-dimensional graded Brauer group are anticipated in the context of Kasparov's $KK$-theory.

This paper presents the results of the author's dissertation, written to complete the requirements for the degree of Ph.D. at Purdue University. She wishes to take this opportunity to express her deep appreciation to Professor J. Kaminker of Indiana University–Purdue University at Indianapolis for his guidance and encouragement. In addition, she would like to thank Professor F. Shultz of Wellesley College for his helpful suggestions.

**1. Preliminaries.** Let $X$ be a locally compact Hausdorff space. The $C^*$-algebra of continuous maps from $X$ to $\mathbf{C}$ which vanish at $\infty$ will be denoted by $C_0(X)$. We will assume that $\mathscr{H}$ is a separable, infinite-dimensional Hilbert space, with inner product denoted by $(\ ,\ )_{\mathscr{H}}$. The group of unitary operators on $\mathscr{H}$, equipped with the strong operator topology, will be denoted by $\mathscr{U}(\mathscr{H})$, and the $C^*$-algebra of bounded operators on $\mathscr{H}$ will be denoted by $\mathscr{L}(\mathscr{H})$. If $I$ is the identity operator, then $S^1$ is included into $\mathscr{U}(\mathscr{H})$ by mapping $s \in S^1$ to $sI \in \mathscr{U}(\mathscr{H})$. The quotient $\mathscr{U}(\mathscr{H})/S^1$ is the projective unitary group of $\mathscr{H}$, denoted by $\mathscr{P}\mathscr{U}(\mathscr{H})$, and is given the quotient topology. Let $\mathscr{K}(\mathscr{H})$ be the $C^*$-algebra of compact operators on $\mathscr{H}$. The automorphism of $\mathscr{K}(\mathscr{H})$, denoted by $\mathrm{Aut}(\mathscr{K})$, will be given the topology of pointwise convergence.

Let $A$ be a $C^*$-algebra. The spectrum of $A$, denoted by $\widehat{A}$, is given the Jacobson topology [**3**, 3.1]. In this paper, attention will be restricted to separable, continuous trace $C^*$-algebras, whose spectra are all Hausdorff [**3**, 4.5.3], and have a countable base [**3**, 3.3.4]. A hermitian element $a \in A$ is called a positive element of $A$ if there exists a $y \in A$ with $y \cdot y^* = a$. Let $A^+$ denote the set of positive elements of $A$. If $t$ is a cardinal, then $A$ is homogeneous of degree $t$ if $\dim(H_\pi) = t$ for every nonzero irreducible representation $\pi$ of $A$.

1.1. DEFINITION. Let $A$ be a $C^*$-algebra with spectrum $X$. Then $A$ is a continuous trace $C^*$-algebra if $X$ is Hausdorff, and if, for every $x \in X$, there is an element $a \in A^+$ and a neighborhood $V_x$ of $x$ in $X$ such that $v(a)$ is a rank one projection for every $v \in V_x$.

This definition is equivalent to the standard one of a continuous trace $C^*$-algebra [**3**, 4.5.3, 4.5.4]. The above characterization will be especially useful here.

1.2. DEFINITION. Suppose that $X$ is a locally compact Hausdorff space. Let $\xi$ be a family of $C^*$-algebras $\{\xi(x)\}_{x\in X}$ together with a set of maps from $X$ to $\bigcup_{x\in X}\xi(x)$, called sections and denoted by $\Gamma(\xi)$, such that

(i) the set of sections forms a $*$-algebra under pointwise operations;
(ii) the set $\{s(x) \colon s \in \Gamma(\xi),\ x \in X\}$ is dense in $\xi(x)$;
(iii) the mapping $s \mapsto ||s(x)||$ is continuous for every $s \in \Gamma(\xi)$;
(iv) if $s : X \to \bigcup_{x\in X}\xi(x)$, then $s \in \Gamma(\xi)$ if, for every $x \in X$ and $\varepsilon > 0$, there is an $s' \in \Gamma(\xi)$ and a neighborhood $V$ of $x$ in $X$ such that $||s(y) - s'(y)|| < \varepsilon$ for all $y$ in $V$.

Then $\xi$ is called a continuous field of $C^*$-algebra over $X$ [**3**, 10.1.2, **21**, 1.3].

Let $E(\xi) = \bigcup_{x\in X}\xi(x)$ be the total space of $\xi$. If $p\colon E(\xi) \to X$ by $p(y) = x$ for $y \in \xi(x)$, then $E(\xi)$ can be equipped with the tube topology [**5**, 1.2]. The set of sections of $\xi$ which vanish at $\infty$, denoted by $\Gamma_0(\xi)$, forms a $C^*$-algebra $A$ with the norm defined by $||s|| = \sup_{x\in X}||s(x)||$ for $s \in \Gamma_0(\xi)$. Its spectrum $\widehat{A}$ is the space $X$ [**3**, 10.4.1], and we can then consider an element of $X$ to be an irreducible representation [**3**, 10.4.4]. Let $\xi$ and $\xi'$ be two continuous fields of $C^*$-algebras over $X$. A function $\varphi\colon \xi \to \xi'$ is an isomorphism if $\varphi$ is the union $\bigcup_{x\in X}\varphi_x$ of isomorphisms $\varphi_x\colon \xi(x) \to \xi'(x)$ such that $\varphi(\Gamma(\xi)) = \Gamma(\xi')$. A continuous field of Hilbert spaces may be defined in a manner similar to the definition of a continuous field of $C^*$-algebras, where each $\xi(x)$ is a now separable Hilbert space [**3**, 10.1.2].

We recall some elementary sheaf theory. The references [**27** and **2**] provide more detail. Let $X$ be a paracompact Hausdorff space. If $G$ is an abelian group, then $\check{H}^*(X;\underline{G})$ is the Čech cohomology of $X$ with coefficients in $\underline{G}$, the sheaf of germs of continuous functions from $X$ into $G$. If $G$ is nonabelian, then the cohomology set $\check{H}^1(X;\underline{G})$ can be defined [**13**, p. 38]. Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be a short exact sequence of groups such that $G_1$ is contained in the center of $G_2$; we can then construct the following exact sequence [**10**]:

$$\cdots \to \check{H}^1(X;\underline{G_1}) \to \check{H}^1(X;\underline{G_2}) \to \check{H}^1(X;\underline{G_3}) \to \check{H}^2(X;\underline{G_1}).$$

1.3. DEFINITION. Let $A$ be a $C^*$-algebra. Then $A$ is a ($\mathbf{Z}_2$-) graded $C^*$-algebra if $A$ can be expressed as the direct sum $A^{(0)} \oplus A^{(1)}$, where $A^{(i)}$, $i = 0, 1$, are selfadjoint, closed linear subspaces of $A$, closed under $^*$, and such that if $a_i \in A^{(i)}$, $a_j \in A^{(j)}$, then $a_i a_j \in A^{(i+j)}$, where addition is modulo 2. If $a \in A^{(i)}$, then $a$ is said to have degree $i$.

Alternatively, a grading on $A$ may be induced from an automorphism $\alpha$ of order 2 on $A$ in the following way. Let $A^{(i)} = \{a \in A\colon \alpha(a) = (-1)^i a\}$, $i = 0, 1$. Then $A^{(0)} \oplus A^{(1)} = A$ is a grading for $A$. If a grading for $A$ is given, the automorphism $\alpha$ can be defined by $\alpha(a_0 + a_1) = a_0 + (-a_1)$. An element $a$ of a graded $C^*$-algebra $A$ is called homogeneous if $a \in A^{(i)}$. A $C^*$-algebra $A$ is trivially graded if $A^{(0)} = A$ and $A^{(1)} = 0$. If $A$ and $B$ are two graded $C^*$-algebras, a $^*$-homomorphism $\psi\colon A \to B$ is graded if $\psi(A^{(i)}) \subset B^{(i)}$.

The grading of $\mathscr{K}(\mathscr{H})$ will now be constructed; the resulting graded $C^*$-algebra will be denoted by $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$. First, we will say that $\mathscr{H}$ is a graded Hilbert space, if it is graded in the following way. Suppose that $\mathscr{H}^{(0)}$ and $\mathscr{H}^{(1)}$ are two copies of $\mathscr{H}$. Since there is an isomorphism $\mathscr{H} \approx \mathscr{H} \oplus \mathscr{H}$, we may write $\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$. An alternate grading on $\mathscr{H}$ uses a unitary operator $J$ of $\mathscr{H}$ with $J^2 = 1$: define $\mathscr{H}^{(i)} = \{h \in \mathscr{H}\colon J(h) = (-1)^i h\}$, where we consider only those operators $J$ for which $\mathscr{H}^{(i)}$ is infinite-dimensional. A direct computation verifies that, if $J$ and $J'$ are two unitary operators of order 2 of a graded Hilbert space $\mathscr{H}$ which determine the same grading of $\mathscr{H}$, then $J = J'$.

An operator $T$ on $\mathscr{H}$ is said to be of degree $i$, $i = 0, 1$, if $T(\mathscr{H}^{(j)}) \subset \mathscr{H}^{(i+j)}$, for $j = 0, 1$. Define a grading for $\mathscr{L}(\mathscr{H})$ by letting $\mathscr{L}^{(i)}(\mathscr{H})$ be the set of bounded operators of degree $i$. For convenience, a matrix is often used to describe a graded operator. A degree 0 operator can be represented by a matrix of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$, where $A\colon \mathscr{H}^{(0)} \to \mathscr{H}^{(0)}$ and $D\colon \mathscr{H}^{(1)} \to \mathscr{H}^{(1)}$. Similarly, a degree 1 operator can be written as $\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$ with $B\colon \mathscr{H}^{(1)} \to \mathscr{H}^{(0)}$ and $C\colon \mathscr{H}^{(0)} \to \mathscr{H}^{(1)}$.

The compact operators on a graded Hilbert space $\mathscr{H}$ can be graded by defining $\mathscr{K}_{\mathrm{gr}}^{(i)}(\mathscr{H})$ to be the compact operators of $\mathscr{H}$ of degree $i$. A unitary $J$ of order 2 on $\mathscr{H}$ may also be used to define the grading on $\mathscr{K}(\mathscr{H})$ (respectively $\mathscr{L}(\mathscr{H})$); let $T \in \mathscr{K}_{\mathrm{gr}}^{(i)}(\mathscr{H})$ (respectively $\mathscr{L}^{(i)}(\mathscr{H})$) if $JTJ^{-1} = (-1)^i T$, for $i = 0, 1$. We can easily check that if $J, J' \in \mathscr{U}(\mathscr{H})$ are of order 2 and induce the same grading on $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$, then $J = \pm J'$. A graded elementary $C^*$-algebra is a graded $C^*$-algebra which is isomorphic to $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$, for $\mathscr{H}$ a graded Hilbert space. The spectrum of a graded $C^*$-algebra $A$ is the usual spectrum of $A$ regarded as an ungraded algebra.

We now define the graded tensor product of $A$ and $B$ [**24**, p. 61; **18**, 2.6]. Let $A$ be a graded $C^*$-algebra. A graded state on $A$ is a positive linear functional $s$ defined on $A$ such that $||s|| = 1$ and $s = 0$ on $A^{(1)}$. If $A$ and $B$ are separable, graded continuous trace $C^*$-algebras, let $A \hat{\odot} B$ denote the algebraic graded tensor product of $A$ and $B$, where the elements of $A \hat{\odot} B$ are graded by

$$\deg(a \hat{\odot} b) = \deg(a) + \deg(b).$$

The product and involution are defined by

$$(a \hat{\odot} b)(a' \hat{\odot} b') = (-1)^{\deg(b)\deg(a')}(aa' \hat{\odot} bb'),$$
$$(a \hat{\odot} b)^* = (-1)^{\deg(a)\deg(b)}(a^* \hat{\odot} b^*).$$

If $s$ and $t$ are graded states on $A$ and $B$, respectively, let

$$s\hat{\odot}t(x^*x) = \sum_{i,j=1}^{n} s(a_i^*a_j)t(b_i^*b_j)$$

for $x = \sum_1^n a_j\hat{\odot}b_j \in A\hat{\odot}B$. Then a $C^*$-norm may be defined on $A\hat{\odot}B$ by

$$||x||_*^2 = \sup_{s,t,y} \frac{s\hat{\odot}t(y^*x^*xy)}{s\hat{\odot}t(y^*y)}$$

where the supremum is taken over all graded states $s$ on $A$, $t$ on $B$, and over all $y \in A\hat{\odot}B$ with $s\hat{\odot}t(y^*y) \neq 0$. Let $A\hat{\otimes}B$ denote the completion of $A\hat{\odot}B$ with respect to the norm $||\ \ ||_*$.

Note that $A\hat{\otimes}B$ defined above is the graded analogue of the minimal tensor product of $A$ and $B$. In the case considered here, $A$ and $B$ are continuous trace, so $A\hat{\otimes}B$ agrees with the graded version of the maximal tensor product [**1**, 16.4]. Thus there is no ambiguity when we refer to the graded tensor product $A\hat{\otimes}B$.

We say that a graded $C^*$-algebra $A$ is stable if $A \approx A\hat{\otimes}\mathscr{K}_{\text{gr}}(\mathscr{H})$, via a graded $^*$-isomorphism. Let $X$ be a locally compact Hausdorff space, with countable base. Then we define $\mathscr{G}(X)$ to be the category whose objects are separable, graded, stable, $C^*$-algebras with continuous trace, with spectrum $X$. We note that the grading of $A$ must be nontrivial; in addition, we require that the grading automorphism $\alpha$ of $A$ fix $X$. It is useful to observe that every element of $\mathscr{G}(X)$ is homogeneous of degree $\aleph_0$ [**21**, 1.12]. A morphism of $\mathscr{G}(X)$ is a graded $^*$-homomorphism. Let $\text{GBr}^\infty(X)$ denote the set of graded isomorphism classes of elements of $\mathscr{G}(X)$.

Let $\xi$ be a fiber bundle over $X$ with fiber $F$ a $C^*$-algebra, and group $G$. Then $\xi$ is a graded fiber bundle if $F = F^{(0)} \oplus F^{(1)}$ is a graded $C^*$-algebra and if the group $G$ is contained in the subgroup of $\text{Aut}(F)$ whose elements preserve the grading of $F$. We note that the local trivializations $h_i\colon \mathscr{U}_i \times F \to \xi|_{\mathscr{U}_i}$, for $\{\mathscr{U}_i\}_{i\in I}$ an open cover of $X$, must preserve the grading on the fiber. In addition, $\xi$ may be written as the Whitney sum $\xi = \xi^{(0)} \oplus \xi^{(1)}$. One example of a graded fiber bundle is a Clifford algebra bundle. If $\xi$ is a real vector bundle over $X$ with a Riemannian metric, then the complexified Clifford algebra bundle of $\xi$, denoted by $C(\xi)$, is a bundle of graded $C^*$-algebras such that $C(\xi)_x = C(F_x) \otimes_{\mathbf{R}} \mathbf{C}$, where $C(F_x)$ is the Clifford algebra associated to the fiber over $x$. Let $\xi$ be an ungraded fiber bundle with fiber $F$ a $C^*$-algebra. Then $\xi$ may be given a trivial grading corresponding to the trivial grading of the fiber $F$. In this case, $\xi^{(0)} = \xi$, and $\xi^{(1)} = 0$. If $\xi$ is a graded fiber bundle over $x$, then $\Gamma_0(\xi)$, the algebra of sections of $\xi$ which vanish at $\infty$, is graded as follows: for $s \in \Gamma_0(\xi)$, $\deg(s) = i$ if $s(x) \in F_x^{(i)}$ for every $x \in X$. If $\xi_1$ and $\xi_2$ are graded fiber bundles, then $\varphi\colon \xi_1 \to \xi_2$ is a graded homomorphism of graded fiber bundles if $\varphi$ is a homomorphism of fiber bundles which preserves the grading on each fiber.

**2. Construction of the fiber bundle associated to a graded $C^*$-algebra.** The aim of this section is to identify each element of $\text{GBr}^\infty(X)$ with one of a Čech cohomology group. Then the powerful techniques of cohomology theory can be used to analyze $\text{GBr}^\infty(X)$. The key step in this identification is the construction of a continuous field of graded $C^*$-algebras from an element of $\mathscr{G}(X)$. This continuous field is then shown to be a fiber bundle. Before proceeding to the actual construction, it is necessary to make some remarks concerning graded representations of a graded $C^*$-algebra.

Let $A \in \mathscr{G}(X)$, and suppose that $\pi\colon A \to \mathscr{L}(\mathscr{H}_\pi)$ is a representation of $A$. Then $\pi$ is a graded representation if $\mathscr{H}_\pi$ is a separable, graded, infinite-dimensional Hilbert space, and $\pi$ is a graded $^*$-homomorphism. As in the ungraded case, a subspace $K$ of a graded Hilbert space $\mathscr{H}$ is said to be invariant under a graded representation $\pi\colon A \to \mathscr{L}(\mathscr{H})$ if $\pi(A)K \subset K$. An irreducible graded representation $\pi$ of $A \in \mathscr{G}(X)$ is a graded representation such that if $K$ is an invariant subspace of $\pi$, then $K = 0$ or $\mathscr{H}$. The quotient $A/\ker(\pi)$ is graded in the following way. Let $a \in A$ be a homogeneous element of $A$. Let $[a]$ be the equivalence class of $a$ in $A/\ker(\pi)$. Define $\deg([a]) = \deg(a)$. Since $\pi$ is graded, this definition is well defined. It then follows that the quotient map $q\colon A \to A/\ker(\pi)$ is graded, and that the homomorphism $\varphi\colon A/\ker(\pi) \to \mathscr{K}_{\mathrm{gr}}(\mathscr{H}_\pi)$ is a graded isomorphism.

2.1. LEMMA. *Let $A \in \mathscr{G}(X)$. Every element $x \in X$ can be represented by a nontrivial irreducible graded representation.*

PROOF. Let $x \in X$ and let $\pi\colon A \to \mathscr{L}(\mathscr{H}_\pi)$ be a representative of the equivalence class $x$. Suppose that $\alpha$ is the grading automorphism of $A$; then $\alpha$ preserves the kernel of $\pi$. Since $A/\ker(\pi) \approx \mathscr{K}_{\mathrm{gr}}(\mathscr{H}_\pi)$, $\alpha$ induces the standard grading on $\mathscr{K}(\mathscr{H}_\pi)$. There exists a $J \in \mathscr{U}(\mathscr{H}_\pi)$ which induces this grading on $\mathscr{K}(\mathscr{H}_\pi)$. Use $J$ to define a grading on $\mathscr{H}_\pi$ as in §1. Then $\pi'\colon A \to \mathscr{L}(\mathscr{H}_\pi^{(0)} \oplus \mathscr{H}_\pi^{(1)})$ by $\pi'(a) = \pi(a)$ is a graded representation of $A$. Since $\pi$ is irreducible, $\pi'$ is also irreducible. And $\ker(\pi) = \ker(\pi')$ implies that $\pi$ and $\pi'$ determine the same equivalence class of $x$. □

It is now possible to construct the continuous field of graded elementary $C^*$-algebras associated to an element of $\mathscr{G}(X)$. Let $A \in \mathscr{G}(X)$. Every $x \in X$ may be identified with an irreducible representation of $A$ on a graded Hilbert space $\mathscr{H}$, and this representation is graded by the above lemma. Then the continuous field $\xi_A$ is the family of $C^*$-algebras $\{\xi(x)\}_{x\in X}$, where $\xi(x) = A/\ker(x)$, together with the set of sections $\Gamma(\xi_A)$ defined as follows. For every $a \in A$, let $s_a : x \mapsto a_x$, where $a_x$ denotes the image of $a$ in $A/\ker(x)$. Let $\mathscr{S} = \{s_a\colon a \in A\}$. Then $\Gamma(\xi_A)$ is the set of maps $s'\colon X \to \bigcup_{x\in X} \xi(x)$ with the property: for every $\varepsilon > 0$ and every $x \in X$, there exists a neighborhood $V$ of $x$ in $X$ and a map $s \in \mathscr{S}$ such that $\|s(y) - s'(y)\| < \varepsilon$ for every $y \in V$. Note that since $A$ is homogeneous of degree $\aleph_0$, then for every $x \in X$, $A/\ker(x)$ is isomorphic to $\mathscr{K}(\mathscr{H})$. Since $A$ is graded, each $\xi(x)$ is graded and the isomorphism between $A/\ker(x)$ and $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ preserves the grading of $\xi(x)$ induced from $A$. This construction is the graded analogue of the Dixmier-Douady construction [**3**, 10.5].

We can proceed now to show that the continuous field $\xi_A$ is a graded fiber bundle, with base $X$ and fiber $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$. First, it is necessary to identify the group of the proposed fiber bundle. Let $\mathrm{Aut}^0(\mathscr{K})$ be the subgroup of $\mathrm{Aut}(\mathscr{K})$ whose elements preserve the grading of $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$. Then $\mathrm{Aut}^0(\mathscr{K})$ inherits the topology of pointwise convergence from $\mathrm{Aut}(\mathscr{K})$. Let

$$\mathscr{U}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathscr{U}(\mathscr{H}) \right\}, \quad \mathscr{U}_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathscr{U}(\mathscr{H}) \right\}.$$

Let $\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) = \mathscr{U}_0 \cup \mathscr{U}_1$; $\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ is a closed subgroup of $\mathscr{U}(\mathscr{H})$ which inherits the strong operator topology from $\mathscr{U}(\mathscr{H})$. Define $\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ to be the quotient $\mathscr{U}_{\mathrm{gr}}(\mathscr{H})/S^1$.

2.2. PROPOSITION. $\mathrm{Aut}^0(\mathscr{H})$ *is homeomorphic to* $\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$.

PROOF. Define a function $\varphi\colon \mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \to \mathrm{Aut}^0(\mathscr{K})$ by $\varphi(U)(T) = UTU^*$, for $U \in \mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ and $T \in \mathscr{K}$. It is clear that the kernel of $\varphi$ is $S^1$, and that $\varphi(\mathscr{U}) \in \mathrm{Aut}^0(\mathscr{K})$ for every $\mathscr{U} \in \mathscr{U}_{\mathrm{gr}}(\mathscr{H})$. We next show that $\varphi$ is surjective. Suppose that $\Phi \in \mathrm{Aut}^0(\mathscr{K})$. There exists a unitary $U$ such that $\Phi(T) = UTU^*$ for every $T \in \mathscr{K}(\mathscr{H})$, and in particular, for every $T \in \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$. Since $\Phi \in \mathrm{Aut}^0(\mathscr{K})$, then $\deg(T) = i$ implies that $\deg(\Phi(T)) = i$, $i = 0, 1$. It can be shown that $U$ is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $a, b, c, d \in \mathscr{U}(\mathscr{H})$, by choosing an orthonormal basis for $\mathscr{H}$, making some appropriate choices for $T$, and then computing $UTU^*$ for these cases.

Using the definition of the strong operator topology, it is easy to show that the map $\varphi\colon \mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \to \mathrm{Aut}^0(\mathscr{K})$ is continuous. Therefore, the quotient map $\bar{\varphi}\colon \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \to \mathrm{Aut}^0(\mathscr{K})$ is bijective and continuous. To complete the proof that $\bar{\varphi}$ is a homeomorphism, it can be shown, following the argument of [**7**, 5.40], that $\bar{\varphi}^{-1}$ is continuous. □

2.3. THEOREM. *Let* $A \in \mathscr{G}(X)$. *Then* $\xi_A$ *is a graded fiber bundle with base space* $X$, *fiber* $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$, *and group* $\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$.

PROOF. The construction above of $\xi_A$ gives the base space $X$, the fiber $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$, and the total space $E(\xi_A) = \bigcup_{x \in X} A/\ker(x)$, which is equipped with the tube topology. Let $p\colon E(\xi_A) \to X$ by $p(y) = x$ when $y \in A_x$. It is straightforward to check that $\mathrm{Aut}^0(\mathscr{K}) \approx \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ is an effective topological transformation group for $\xi_A$. The rest of the defining conditions for a fiber bundle are satisfied by the following proposition.

2.4. PROPOSITION. *There exist coordinate neighborhoods* $\{\mathscr{U}_i\}_{i \in I}$ *of* $X$ *and graded homeomorphisms* $h_i\colon \mathscr{U}_i \times \mathscr{K}_{\mathrm{gr}}(\mathscr{H}) \to p^{-1}(\mathscr{U}_i)$ *which satisfy*

(i) $ph_i(x, T) = x$, *for every* $x \in \mathscr{U}_i$, $T \in \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$;

(ii) *if* $h_{i,x}\colon \mathscr{K}_{\mathrm{gr}}(\mathscr{H}) \to p^{-1}(x)$ *is defined by setting* $h_{i,x}(T) = h_i(x, T)$, *then, for each pair* $i, j \in I$, *and each* $x \in \mathscr{U}_i \cap \mathscr{U}_j$, *the homeomorphism* $h_{i,x}^{-1} \circ h_{j,x}\colon \mathscr{K}_{\mathrm{gr}}(\mathscr{H}) \to \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ *coincides with an element of* $\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$;

(iii) *for each* $i, j \in I$, *the map* $g_{i,j}\colon \mathscr{U}_i \cap \mathscr{U}_j \to \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ *defined by* $g_{i,j}(x) = h_{i,x}^{-1} \circ h_{j,x}$ *is continuous.*

The proof of Proposition 2.4 will be delayed until §6. This will conclude the proof that $\xi_A$ is a graded fiber bundle. □

**3.** $\mathrm{GBr}^\infty(X) \approx \check{H}^1(X; \underline{\mathbf{Z}}_2) \oplus \check{H}^3(X; \underline{\mathbf{Z}})$. We now prove that $\mathrm{GBr}^\infty(X)$ is isomorphic to $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$, which in turn is isomorphic to $\check{H}^1(X; \underline{\mathbf{Z}}_2) \oplus \check{H}^3(X; \underline{\mathbf{Z}})$, and discuss the group structure of each. It is first shown that $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ and $\mathrm{GBr}^\infty(X)$ are isomorphic, as sets. Let $\mathscr{B}(X)$ be the category whose objects are graded fiber bundles over $X$, with fiber $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ and group $\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$. A morphism between objects of $\mathscr{B}(X)$ is a graded homomorphism of graded fiber bundles.

Then $\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ can be regarded as the set formed from the graded isomorphism classes of elements of $\mathscr{B}(X)$. Let $\xi \in \mathscr{B}(X)$. Define the functions $\tau$ and $\tau'$ as follows:

$$\begin{aligned} \tau &\colon \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \to \mathrm{GBr}^\infty(X) \quad \text{by } \tau([\xi]) = [\Gamma_0(\xi)], \\ \tau' &\colon \mathrm{GBr}^\infty(X) \to \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \quad \text{by } \tau'([A]) = [\xi_A]. \end{aligned}$$

It will be shown that $\tau$ and $\tau'$ are well-defined natural functions, such that $\tau'$ is inverse to $\tau$.

The following proposition verifies that $\tau$ and $\tau'$ are well defined.

3.1. PROPOSITION. (i) *If $\xi$ and $\xi' \in \mathscr{B}(X)$ such that $[\xi] = [\xi']$ in $\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$, then $[\Gamma_0(\xi)] = [\Gamma_0(\xi')]$ in $\mathrm{GBr}^\infty(X)$.*

(ii) *If $A$ and $B \in \mathscr{G}(X)$ such that $[A] = [B]$ in $\mathrm{GBr}^\infty(X)$, then $[\xi_A] = [\xi_B]$ in $\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$.*

PROOF. (i) If $f\colon E(\xi) \to B(\xi')$ is a graded, fiber-preserving isomorphism, it is easy to verify that $\Gamma_0(f)$ is a graded isomorphism from $\Gamma_0(\xi)$ to $\Gamma_0(\xi')$.

(ii) Suppose that $\varphi\colon A \to B$ is a graded $^*$-isomorphism. Let $x \in X$ correspond to $\ker(\pi)$, where $\pi$ is an irreducible graded representation of $A$. Let $\pi' = \pi\varphi^{-1}\colon B \to \mathscr{L}(\mathscr{H})$. Consider the following diagram, which defines $\bar{\varphi}_x$.

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & B \\ {\scriptstyle q_A}\big\downarrow & & \big\downarrow{\scriptstyle q_B} \\ A/\ker(\pi) & \xrightarrow{\ \bar{\varphi}_x\ } & B/\ker(\pi') \end{array}$$

Note that $\bar{\varphi}_x$ is a graded isomorphism for each $x \in X$. Hence $\varphi_x$ is a graded isomorphism from each fiber of $\xi_B$. Let $\Phi = \bigcup_{x\in X} \bar{\varphi}_x$. Then $\Phi$ is a graded isomorphism from $\xi_A$ to $\xi_B$. $\square$

The next proposition verifies that $\tau'$ is inverse to $\tau$.

3.2. PROPOSITION. *Let $A \in \mathscr{G}(X)$ and $\xi \in \mathscr{B}(X)$. Then*

(i) *$A$ and $\Gamma_0(\xi_A)$ are isomorphic as graded $C^*$-algebras;*

(ii) *$\xi$ and $\xi_{\Gamma_0(\xi)}$ are isomorphic as graded fiber bundles.*

PROOF. (i) By [**3**, 10.5.4], there is an isomorphism which maps an element $a \in A$ to the section $s_a$ of $\xi_A$ defined by $s_a(x) = a_x$, for $x \in X$, where $a_x$ is the image of $a$ in $A/x$. Since the projection $a\colon A \to A/x$ preserves the grading, the isomorphism $a \mapsto s_a$ preserves the grading.

(ii) Let $y_x \in \mathscr{K}_{\mathrm{gr}}(\mathscr{H}) = \xi_x$. There is a section $s\colon X \to E(\xi)$ by $s(x) = y_x$ for every $x \in X$. Let $q_x\colon \Gamma_0(\xi) \to \Gamma_0(\xi)/x$ be the quotient map, and let $s_x$ denote the image of $s$ under $q_x$. The canonical isomorphism between $\xi_x$ and $\Gamma_0(\xi)/x$ is then defined by $y_x \mapsto s_x$ [**3**, 10.5.2]. This isomorphism is graded on each fiber since $q_x$ preserves the grading. Hence $\xi$ and $\xi_{\Gamma_0(\xi)}$ are isomorphic as graded fiber bundles. $\square$

Therefore, there is a one-to-one correspondence between $\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ and $\mathrm{GBr}^\infty(X)$. Before proceeding to the discussion of their operations, it will be shown that $\tau$ is natural. Suppose $f\colon X \to Y$ is a map, where $X$ and $Y$ are locally compact

Hausdorff spaces, each with countable base. Let $\xi \in \mathscr{B}(Y)$ and $B \in \mathscr{G}(Y)$. Then $f$ induces the functions

$$f^*\colon \check{H}^1(Y; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \to \check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \quad \text{by } [\xi] \mapsto [f^*(\xi)]$$

and

$$\bar{f}\colon \mathrm{GBr}^\infty(Y) \to \mathrm{GBr}^\infty(X) \quad \text{by } [\Gamma_0(\xi_B)] \mapsto [\Gamma_0(f^*\xi_B)].$$

Then the following diagram commutes:

$$\begin{array}{ccc} \check{H}^1(Y; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) & \xrightarrow{f^*} & \check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \\ \tau\downarrow & & \downarrow\tau \\ \mathrm{GBr}^\infty(Y) & \xrightarrow{\bar{f}} & \mathrm{GBr}^\infty(X). \end{array}$$

Next, the operations for $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ and $\mathrm{GBr}^\infty(X)$ are discussed. In addition, it is shown that $\tau$ and $\tau'$ respect these operations. The fiberwise graded tensor product of graded fiber bundles is the operation of $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$. Specifically, if $\xi, \xi' \in \mathscr{B}(X)$, let $[\xi]\hat{\otimes}_X[\xi'] = [\xi\hat{\otimes}_X\xi']$. This fiberwise tensor product on infinite-dimensional bundles must be carefully defined; see [**9**, p. 78] for a more complete discussion of the ungraded case. Let $\xi_0$ denote the trivial bundle over $X$ with fiber $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$. Then the identity element of $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ is $[\xi_0]$.

Let $A, B \in \mathscr{G}(X)$. Then $A$ and $B$ are $C_0(X)$-modules, and we define $[A]\hat{\otimes}_X[B] = [A\hat{\otimes}_{C_0(X)}B]$. Note that the operation $\hat{\otimes}_{C_0(X)}$ is not the usual algebraic tensor product, but a graded version of a $C^*$-algebraic construction due to Rieffel and Green [**11**]. By Propositions 3.1 and 3.2, $[A]\hat{\otimes}_X[B] = [\Gamma_0(\xi_A\hat{\otimes}_X\xi_B)]$. It is clear that the identity element of $\mathrm{GBr}^\infty(X)$ is the equivalence class of the $C^*$-algebra of maps from $X$ to $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ which vanish at $\infty$. It is immediate that $\tau([\xi_0]) = 1_{\mathrm{GBr}^\infty(X)}$. We have, for $\xi, \xi' \in \mathscr{B}(X)$, that

$$\tau([\xi]\hat{\otimes}_X[\xi']) = \tau([\xi])\hat{\otimes}_X\tau([\xi']).$$

We now can proceed to the definition of the function $w$. Let $w_1\colon \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \to \mathbf{Z}_2$ be defined by $w_1([a]) = (-1)^{\deg(a)}$. It is easy to check that $w_1$ is well defined. Recall that the Bockstein homomorphism $\delta_j^*\colon \check{H}^j(X; \underline{S^1}) \to \check{H}^{j+1}(X; \underline{\mathbf{Z}})$ associated to the exact sequence $1 \to \mathbf{Z} \to \mathbf{R} \to S^1 \to 1$ is an isomorphism. The short exact sequence $1 \to S^1 \to \mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \to \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \to 1$ induces the following exact sequence

$$\text{(I)} \quad \cdots \to \check{H}^1(X; \underline{S^1}) \to \check{H}^1(X; \underline{\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \to \check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \xrightarrow{\tilde{\delta}_1^*} \check{H}^2(X; \underline{S^1}).$$

Let $w_2^* = \delta_2^*\tilde{\delta}_1^*$. Define

$$w\colon \check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \to \check{H}^1(X; \underline{\mathbf{Z}_2}) \oplus \check{H}^3(X; \underline{\mathbf{Z}}) \quad \text{by } w(x) = (w_1^*(x), w_2^*(x)).$$

Using the exactness of (I) and the definition of $w_1$, it is straightforward to verify the following lemma.

3.3. LEMMA. *Let $x \in \check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$. Then $w(x) = (1, 0)$ implies that $x$ is the identity element in $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$.*

3.4. PROPOSITION. *Let $\xi, \xi' \in \mathscr{B}(X)$, and let $\beta$ be the Bockstein homomorphism associated to the sequence $1 \longrightarrow \mathbf{Z} \xrightarrow{(\times 2)} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_2 \longrightarrow 1$, where $r(n) = (-1)^n$. Then*

$$w([\xi \hat{\otimes}_X \xi']) = (w_1^*([\xi]) \cdot w_1^*([\xi']), w_2^*([\xi]) + w_2^*([\xi']) + \beta(w_1^*([\xi']) \cup w_1^*([\xi]))).$$

The proof parallels that of [**6**, Lemma 10] and will be omitted.

An explicit inverse to an arbitrary element of $\mathrm{GBr}^\infty(X)$ will now be given. Let $A \in \mathscr{G}(X)$ and let $\xi_A$ be the graded fiber bundle associated to $A$. Let $\bar{\xi}_A$ be the fiber bundle which is topologically identical to $\xi_A$, and where the elements in each fiber have the same grading as the corresponding ones of $\xi_A$. The fiber of $\bar{\xi}_A$ is $\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$; let $\bar{\xi}_A$ have the following fiberwise operations, for every $x, y \in \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$, $c \in \mathscr{C}$:

| | |
|---|---|
| addition: | $(x, y) \mapsto x + y$ |
| scalar multiplication: | $(c, x) \mapsto \bar{c}x$ |
| multiplication: | $(x, y) \mapsto (-1)^{\deg(x)\deg(y)} xy$ |
| involution: | $x \mapsto x^*$ |
| norm: | $x \mapsto \|\|x\|\|$ |

Denote the new multiplication by $x \times y$.

3.5. PROPOSITION. *Let $A \in \mathscr{G}(X)$. Then $[\bar{\xi}_A]$ is inverse to $[\xi_A]$ in*

$$\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}).$$

PROOF. By Lemma 3.4, it is sufficient to show that $w([\xi_A \hat{\otimes}_X \bar{\xi}_A]) = (1, 0)$. Let $d_{ij}$ be the transition functions for $\xi_A$, $i, j \in I$. Then the transition functions for $\bar{\xi}_A$ are also $d_{ij}$. Hence $w_1^*([\xi_A]) = w_1^*([\bar{\xi}_A])$, so $w_1^*([\xi_A]) \cdot w_1^*([\bar{\xi}_A]) = 1$.

To calculate $w_2^*([\xi_A \hat{\otimes}_X \bar{\xi}_A])$, we need to do the following computation. Let $g_{ij}$ (respectively $g'_{ij}$) be the element of $\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ which implements the transition function $d_{ij}$ for $\xi_A$ ($d'_{ij}$ for $\bar{\xi}_A$). Let $g_{ij}g_{jk} = u_{ijk}g_{ik}$ and $g'_{ij}g'_{jk} = u'_{ijk}g'_{ik}$. Then

$$\begin{aligned}(g_{ij} \hat{\otimes} g'_{ij})(g_{jk} \hat{\otimes} g'_{jk}) &= (-1)^{\deg(g'_{ij})\deg(g_{jk})}(g_{ij}g_{jk}) \hat{\otimes} (g'_{ij} \times g'_{jk})) \\ &= u_{ijk}u'_{ijk}(g_{ik} \hat{\otimes} g'_{ik}),\end{aligned}$$

since $\deg(g_{jk}) + \deg(g'_{jk}) = 0$. Hence $w_2^*([\xi_A \hat{\otimes}_X \bar{\xi}_A]) = w_2^*([\xi_A]) + w_2^*([\bar{\xi}_A])$. But $u'_{ijk} = \bar{u}_{ijk}$, the complex conjugate of $u_{ijk}$. Therefore $w_2^*([\xi_A]) = -w_2^*([\bar{\xi}_A])$, or $w_2^*([\xi_A \hat{\otimes}_X \bar{\xi}_A]) = 0$. $\square$

If $A \in \mathscr{G}(X)$, the inverse element to $[A] \in \mathrm{GBr}^\infty(X)$ is the element $[\Gamma_0(\bar{\xi}_A)]$. This completes the verification that $\check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ and $\mathrm{GBr}^\infty(X)$ are groups, and that the function $\tau\colon \check{H}^1(X; \underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \to \mathrm{GBr}^\infty(X)$ is a group homomorphism.

It is shown below that $w$ is an isomorphism. Let $T = \mathscr{U}_0/S^1$. Let $\eta\colon \mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \to \mathbf{Z}_2$ be defined by $\eta(a) = (-1)^{\deg(a)}$. Then we have the following diagram of short exact sequences of groups, where $\gamma$ and $\tilde{\gamma}$ are inclusions cf. [**6**]:

$$
\text{(II)}\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & & & \downarrow \\
1 & \to & S^1 & \xrightarrow{\tilde{\gamma}} & \mathscr{U}_0 & \to & T & \to & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
1 & \to & S^1 & \xrightarrow{\gamma} & \mathscr{U}_{\mathrm{gr}}(\mathscr{H}) & \xrightarrow{\nu} & \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) & \to & 1 \\
 & & & & \downarrow \eta & & \downarrow w_1 & & \\
 & & & & \mathbf{Z}_2 & = & \mathbf{Z}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Diagram (II) induces
(III)

$$
\begin{array}{ccccccc}
 & \check{H}^1(X;\underline{S}^1) & \xrightarrow{\tilde{\gamma}^*} & \check{H}^1(X;\underline{\mathscr{U}_0}) & & & \\
 & \downarrow = & & \downarrow & & & \\
\cdots \to & \check{H}^1(X;\underline{S}^1) & \xrightarrow{\gamma^*} & \check{H}^1(X;\underline{\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) & \xrightarrow{\nu^*} & \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) & \xrightarrow{\tilde{\delta}_1^*} \check{H}^2(X;\underline{S}^1) \\
 & & & \downarrow \eta^* & \nearrow \tilde{\nu}^* & & \\
 & & & \check{H}^1(X;\underline{\mathbf{Z}_2}) & & &
\end{array}
$$

It is easy to verify that $\check{H}^1(X;\underline{\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ is a group; hence diagram (III) is a commutative diagram of groups. The set $\mathscr{U}_0$ is contractible [**18**], so $\check{H}^1(X;\underline{\mathscr{U}_0}) = 0$ by [**14**], and therefore $\gamma^* = 0$. In addition, $\eta^*$ is injective. Let $\varsigma\colon \mathbf{Z}_2 \to \mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ be defined by $\varsigma(+1) = \left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$ and $\varsigma(-1) = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Then $\eta s = 1_{\mathbf{Z}_2}$, so $\eta^*$ is surjective. Let $\tilde{\nu}^* = \nu^*(\eta^*)^{-1}$. Then diagram (III) reduces to the following exact sequence:

$$0 \to \check{H}^1(X;\underline{\mathbf{Z}_2}) \xrightarrow{\tilde{\nu}^*} \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \xrightarrow{\tilde{\delta}_1^*} \check{H}^2(X;\underline{S}^1).$$

One result of the theorem below is the fact that $\tilde{\delta}_1^*$ is surjective; hence

$$\text{(IV)}\qquad 0 \to \check{H}^1(X;\underline{\mathbf{Z}_2}) \xrightarrow{\tilde{\nu}^*} \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) \xrightarrow{\tilde{\delta}_1^*} \check{H}^2(X;\underline{S}^1) \to 0$$

is exact. It is also shown that the sequence (IV) splits.

3.6. THEOREM. *$w$ is an isomorphism.*

PROOF. It is necessary to show that $\tilde{\delta}_1^*$ is surjective. Let $\theta\colon \mathscr{U}(\mathscr{H}) \to \mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ by $\theta(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, and $\bar{\theta}\colon \mathscr{P}\mathscr{U}(\mathscr{H}) \to \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ by $\bar{\theta}([a]) = [\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)]$. It is easy to check that $\bar{\theta}$ is well defined. Let $\xi_2$ be the trivial bundle over $X$ with fiber $M = M_2(\mathbf{C})$. The grading of $M$ is defined by

$$M^{(0)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbf{C} \right\} \quad \text{and} \quad M^{(1)} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathbf{C} \right\}.$$

Note that $\bar{\theta}^*([\xi]) = [\xi \hat{\otimes}_X \xi_2]$. We have the following commutative diagram, where the sequences are exact:

$$\text{(V)}\qquad \begin{array}{ccccccccc} 1 & \to & S^1 & \to & \mathscr{U}_{\mathrm{gr}}(\mathscr{H}) & \to & \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) & \to & 1 \\ & & \downarrow{=} & & \downarrow{\theta} & & \downarrow{\bar{\theta}} & & \\ 1 & \to & S^1 & \to & \mathscr{U}(\mathscr{H}) & \to & \mathscr{P}\mathscr{U}(\mathscr{H}) & \to & 1 \end{array}$$

This induces the commutative diagram:

$$\text{(VI)}\qquad \begin{array}{ccccccccc} \cdots \to & \check{H}^1(X;\underline{S}^1) & \to & \check{H}^1(X;\underline{\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) & \to & \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) & \xrightarrow{\tilde{\delta}_1^*} & \check{H}^2(X;\underline{S}^1) \\ & \downarrow{=} & & \downarrow{\theta^*} & & \downarrow{\bar{\theta}^*} & & \downarrow{=} \\ \cdots \to & \check{H}^1(X;\underline{S}^1) & \to & \check{H}^1(X;\underline{\mathscr{U}(\mathscr{H})}) & \to & \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}(\mathscr{H})}) & \xrightarrow{\delta_1^*} & \check{Y}^2(X;\underline{S}^1) \end{array}$$

Since $\delta_1^*$ is an isomorphism, $\tilde{\delta}_1^*$ is surjective. Note that $w_1\tilde{\nu} = 1_{\mathbf{Z}_2}$, so $w_1^*$ is surjective. Hence $w$ is surjective, since both $w_1^*$ and $\delta_2^*\bar{\delta}_1^*$ are. Lemma 3.3 implies that $w$ is injective. □

**4. Interpretations of the invariants $w_1^*$ and $w_2^*$.** Let $A$ be a separable, stable, continuous trace $C^*$-algebra, with spectrum $X$. Then the Dixmier-Douady invariant of $A$, $\delta(A)$, is the image of the fiber bundle constructed from $A$ under the composite

$$\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}(\mathscr{H})}) \xrightarrow{\delta_1^*} \check{H}^2(X;\underline{S}^1) \xrightarrow{\delta_2^*} \check{H}^3(X;\underline{\mathbf{Z}}).$$

Let $\bar{\theta}\colon \mathscr{P}\mathscr{U}(\mathscr{H}) \to \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ be the map defined in the proof of Theorem 3.6. The composite $\mathscr{P}\mathscr{U}(\mathscr{H}) \xrightarrow{\bar{\theta}} \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \xrightarrow{w_1} \mathbf{Z}_2$ maps every element of $\mathscr{P}\mathscr{U}(\mathscr{H})$ to $+1$, so $w_1^*\bar{\theta}^*$ is the zero map. Therefore, it is straightforward to compute the following:

4.1. PROPOSITION. $w(\bar{\theta}^*[\xi_A]) = \delta(A)$.

There is an alternate way to view $w_2^*$. Since $\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) \subset \mathscr{U}(\mathscr{H})$, we can consider the commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \to & S^1 & \to & \mathscr{U}(\mathscr{H}) & \to & \mathscr{P}\mathscr{U}(\mathscr{H}) & \to & 1 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 1 & \to & S^1 & \to & \mathscr{U}_{\mathrm{gr}}(\mathscr{H}) & \to & \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H}) & \to & 1 \end{array}$$

This induces the following diagram:

$$\begin{array}{ccccccccc} \cdots & \to & \check{H}^1(X;\underline{\mathscr{U}(\mathscr{H})}) & \to & \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}(\mathscr{H})}) & \xrightarrow{\delta_1^*} & \check{H}^2(X;\underline{S}^1) & \overset{\delta_2^*}{\approx} & \check{H}^3(X;\underline{\mathbf{Z}}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & \check{H}^1(X;\underline{\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) & \to & \check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})}) & \xrightarrow{\tilde{\delta}_1^*} & \check{H}^2(X;\underline{S}^1) & \overset{\delta_2^*}{\approx} & \check{H}^3(X;\underline{\mathbf{Z}}) \end{array}$$

The homomorphism from $\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})})$ to $\check{H}^1(X;\underline{\mathscr{P}\mathscr{U}(\mathscr{H})})$, which is induced from the inclusion, maps $[\xi]$ to $[\xi^*]$, where $\xi^*$ is the ungraded $\mathscr{P}\mathscr{U}(\mathscr{H})$-bundle underlying $\xi$. We now have the following proposition.

4.2. PROPOSITION. *Let $A \in \mathcal{G}(X)$. Let $\xi_A$ be the $\mathcal{P}\mathcal{U}_{\mathrm{gr}}(\mathcal{H})$-module constructed from A. Let $A^*$ be A considered as an ungraded $C^*$-algebra. Then $\xi_{A^*} = (\xi_A)^*$ is the ungraded $\mathcal{P}\mathcal{U}(\mathcal{H})$-bundle underlying $\xi_A$, and $w_2^*[\xi_A] = \delta(A^*)$.*

The invariant $w_1^*$ measures the grading of the given graded $C^*$-algebra. We have the following characterization.

4.3. PROPOSITION. *Let $A \in \mathcal{G}(X)$. Then $w_1^*[\xi_A] = 1$ if and only if $A \approx A' \hat{\otimes} M_2(\mathbf{C})$, where $A'$ is a separable, stable, continuous trace $C^*$-algebra, with spectrum X, such that $(A')^{(0)} = A'$ and $(A')^{(1)} = 0$.*

PROOF. We have $[A] = [A' \hat{\otimes} M_2(\mathbf{C})]$ if and only if $[\xi_A] = [\xi_{A'} \hat{\otimes}_X \xi_2]$ if and only if $[\xi_A]$ is in the image of $\bar{\theta}^*$ if and only if $w_1^*[\xi_A] = 1$. □

We can also apply the work of J. Philips and I. Raeburn [**21**] to interpret $w_1^*$. Recall that associated to a graded $C^*$-algebra is a grading automorphism of order 2. Suppose that $A$ is a separable, stable, continuous trace $C^*$-algebra, with spectrum $X$. Let $\mathrm{Inn}(A)$ denote the automorphisms of $A$ which are implemented by unitaries in the multiplier algebra, and let $\mathrm{Aut}_{C_0(X)}(A)$ denote the automorphisms of $A$ which fix $C_0(X)$. There is a map $\varphi\colon \mathrm{Aut}_{C_0(X)}(A) \to \check{H}^1(X; \underline{S}^1)$ which fits into the following short exact sequence [**21**, 2.1]:

$$\text{(VII)} \qquad 0 \to \mathrm{Inn}(A) \to \mathrm{Aut}_{C_0(X)}(A) \xrightarrow{\varphi} \check{H}^1(X; \underline{S}^1) \approx \check{H}^2(X; \underline{\mathbf{Z}}) \to 0.$$

Let $\mathcal{H}$ be a Hilbert space and suppose that $J$ is a unitary of degree 2 on $\mathcal{H}$, which is used to define a grading on $\mathcal{H}$. It is straightforward to check that, for $m \in \mathcal{U}_{\mathrm{gr}}(\mathcal{H})$, $w_1^*[m] = (mJ)(Jm)^{-1}$ is a well defined method of computing $w_1^*$. Note that $J$ defines an automorphism of order 2 which gives the grading on each fiber of $\xi_A$. Let $i\colon \mathbf{Z}_2 \to S^1$ be the inclusion, and suppose that $\alpha$ is the automorphism of $A$ which determines the grading of $A$. Using this definition of $w_1^*$, we can calculate that $\varphi(\alpha) = i^* w_1^*([\xi_A])$.

A grading operator of $A$ is a selfadjoint unitary $g$ contained in the multiplier algebra of $A$, such that $A^{(i)} = \{a \in A\colon gag^* = (-1)^i a\}$ for $i = 0, 1$. The short exact sequence (VII) then implies that $w_1^*[\xi_A] = 1$ when the grading of $A$ is determined by a grading operator.

Donovan and Karoubi [**6**] consider the case where $\xi$ is a fiber bundle over a finite complex $X$, with fiber $F$ a simple central graded $\mathbf{C}$-algebra [**30**]. The isomorphism classes of such bundles form a group, $\mathrm{GBr}\, U(X)$. They prove that [**6**, Theorem 11]

$$\mathrm{GBr}\, U(X) \approx \check{H}^0(X; \underline{\mathbf{Z}}_2) \oplus \check{H}^1(X; \underline{\mathbf{Z}}_2) \oplus \mathrm{Tors}(\check{H}^3(X; \underline{\mathbf{Z}})).$$

This isomorphism defines invariants $u_1[\xi] \in \check{H}^1(X; \underline{\mathbf{Z}}_2)$ and $u_2[\xi] \in \mathrm{Tors}(\check{H}^3(X; \underline{\mathbf{Z}}))$ for the element $[\xi] \in \mathrm{GBr}\, U(X)$. Let $\xi_0$ be the trivial bundle over $X$ with fiber $\mathcal{K}_{\mathrm{gr}}(\mathcal{H})$. Given $\xi \in \mathrm{GBr}\, U(X)$, we can include $\xi$ into $\mathrm{GBr}^\infty(X)$ by mapping $[\xi] \to [\xi_0 \hat{\otimes}_X \xi]$. It can then be verified that $w_j^*[\xi] = u_j[\xi]$, $j = 1, 2$. The case where the fiber is a simple central $\mathbf{R}$-algebra [**6**, **19**, **20**, **30**] can be considered by first complexifying the given bundle and then mapping it into $\mathrm{GBr}^\infty(X)$ as above.

Let $V$ be a real $n$-dimensional vector bundle over $X$ with fiber $F$. Suppose that $V$ is equipped with a Riemannian metric. Let $C(V)$ denote the Clifford algebra bundle of $V$, and let $C(V) \otimes_{\mathbf{R}} \mathbf{C}$ denote the complexification of $C(V)$. Let $w_i(V) \in \check{H}^i(X; \underline{\mathbf{Z}}_2)$, $i = 1, 2$, denote the usual Stiefel-Whitney classes of $V$. Let $\beta\colon \check{H}^2(X; \underline{\mathbf{Z}}_2) \to \check{H}^3(X; \underline{\mathbf{Z}})$ be the Bockstein homomorphism associated to the short exact sequence $1 \to \mathbf{Z} \xrightarrow{(\times 2)} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_2 \to 1$. Then, using [**6**, p. 165], we obtain the result that

$$w_1(V) = w_1^*([C(V) \otimes_{\mathbf{R}} \mathbf{C}]) \quad \text{and} \quad \beta w_2(V) = w_2^*([C(V) \otimes_{\mathbf{R}} \mathbf{C}]).$$

**5. Graded Morita equivalence.** Let $A$ be a separable graded continuous trace $C^*$-algebra with spectrum $X$. Then $A\hat{\otimes}\mathscr{K}_{\text{gr}}(\mathscr{H})$ is an element of $\mathscr{G}(X)$. If $B$ is another graded continuous trace $C^*$-algebra, we would like to define an equivalence between $A$ and $B$ which would imply that $[A\hat{\otimes}\mathscr{K}_{\text{gr}}(\mathscr{H})] = [B\hat{\otimes}\mathscr{K}_{\text{gr}}(\mathscr{H})]$ in $\text{GBr}^\infty(X)$. The work in this section determines that the appropriate equivalence is graded Morita equivalence, which is based on the standard definition of strong Morita equivalence. In [**22**], M. Rieffel presented the theory for ungraded $C^*$-algebras.

In an unpublished note [**11**], P. Green gives a variant on the construction of the Dixmier-Douady invariant for ungraded continuous trace $C^*$-algebras. We now consider a graded version of Green's development. Let $A \in \mathscr{G}(X)$. By Lemma 6.2 below, there exists a locally finite open cover $\{\mathscr{U}_i\}_{i\in I}$ of $X$ such that, for every $i \in I$, there exists $a_i \in A^{(0)}$ with $x(a_i)$ a degree 0 rank one projection for all $x \in \mathscr{U}_i$. Let $p_i(x) = x(a_i)$. Suppose $i, j \in I$ and $x \in \mathscr{U}_i \cap \mathscr{U}_j$. Let $\text{Im}(p_i(x)) = \mathbf{C}e_{i,x}$ and $\text{Im}(p_j(x)) = \mathbf{C}e_{j,x}$, for $e_{i,x}$ and $e_{j,x}$ some chosen unit vectors of $\mathscr{H}$. There exists a partial isometry $b \in \mathscr{L}(\mathscr{H})$ whose initial space is $\mathbf{C}e_{j,x}$ and whose range is $\mathbf{C}e_{i,x}$. Let $c \in A$ such that $x(c) = b$. Then $x(a_i c a_j) = x(c) \neq 0$, and in some neighborhood of $x$, $v(a_i c a_j)$ is a rank one operator.

Now replace $\{\mathscr{U}_i\}_{i\in I}$ with a locally finite refinement such that for all $i, j \in I$, there exists $c_{ij} \in A$ with $x(a_i c_{ij} a_j) = x(c_{ij}) \neq 0$ for all $x \in \mathscr{U}_i \cap \mathscr{U}_j$. Let $b_{ij}(x) = x(c_{ij})$. Note that the fact that $x(a_i)$ and $x(a_j)$ are degree 0 projections implies that $b_{ij}(x)$ is a homogeneous operator for every $x \in \mathscr{U}_i \cap \mathscr{U}_j$. Hence $c_{ij}$ is a homogeneous element of $A$. Since $b_{kj}(x)b_{ji}(x)$ and $b_{ki}(x)$ are, for $x \in \mathscr{U}_i \cap \mathscr{U}_j \cap \mathscr{U}_k$, two partial isometries with the same one-dimensional initial space and range, there exists an element $\gamma_{ijk}(x) \in S^1$ such that

$$b_{kj}(x)b_{ji}(x)b_{ki}(x)^* = \gamma_{ijk}(x) \cdot I.$$

The $\{\gamma_{ijk}\}$ form a 2-cocycle in $C^2(X; \underline{S}^1)$. It can be verified that the cohomology class $[\{\gamma_{ijk}\}] \in \check{H}^2(X; \underline{S}^1)$ is independent of the choices made. Let $A \in \mathscr{G}(X)$. Then define $w'(A) = (w_1'(A), w_2'(A))$ where $w_1'(A) = [\{(-1)^{\deg(c_{ji})}\}]$ and $w_2'(A) = \delta_2^*[\{\gamma_{ijk}\}]$. It can be shown that $w'(A) = w[\xi_A]$.

It is appropriate now to turn to a definition of graded Morita equivalence. Let $A$ and $B$ be graded $C^*$-algebras and $M$ a graded left $A$-module and right $B$-module. Then, for $i, j = 0, 1$, one has $A^{(i)}M^{(j)} \subset M^{(i+j)}$ and $M^{(i)}B^{(j)} \subset M^{(i+j)}$. If $A$ is a graded $C^*$-algebra and $M$ a graded $A$-module, an $A$-valued inner product on $M$ is a function $\langle\ ,\ \rangle_A \colon M \times M \to A$ where $\langle M^{(i)}, M^{(j)}\rangle_A \subset A^{(i+j)}$.

5.1 DEFINITION. Two graded $C^*$-algebras $A$ and $B$ are graded Morita equivalent if there exists a graded left-$A$-right-$B$-bimodule $M$ equipped with $A$- and $B$-valued inner products $\langle\ ,\ \rangle_A$ and $\langle\ ,\ \rangle_B$ satisfying:

(a) the requirements for strong Morita equivalence:

(1) $\langle x, x\rangle_A \geq 0$; $\langle x, x\rangle_B \geq 0$;

(2) $\langle x, y\rangle_A^* = \langle y, x\rangle_A$; $\langle x, y\rangle_B^* = \langle y, x\rangle_B$;

(3) $\langle ax, y\rangle_A = a\langle x, y\rangle_A$; $\langle x, yb\rangle_B = \langle x, y\rangle_B b$;

(4) $\langle xb, y\rangle_A = \langle x, yb^*\rangle_A$; $\langle ax, y\rangle_B = \langle x, a^*y\rangle_B$;
(5) $\langle x, y\rangle_A z = x\langle y, z\rangle_B$;
(6) $\langle ax, ax\rangle_B \leq ||a||^2\langle x, x\rangle_B$; $\langle xb, xb\rangle_A \leq ||b||^2\langle x, x\rangle_A$;

for $x, y, z \in M$, $a \in A$, $b \in B$;

(b) the graded requirements:
(1) the span of $\langle M^{(i)}, M^{(j)}\rangle_A$ is dense in $A^{(i+j)}$;
(2) the span of $\langle M^{(i)}, M^{(j)}\rangle_B$ is dense in $B^{(i+j)}$.

$M$ is called a graded $A$-$B$-equivalence bimodule.

Note that if $A$ and $B$ are graded Morita equivalent, they are strong Morita equivalent. The definition of graded Morita equivalence is justified by the following proposition.

5.2. PROPOSITION. *Let $A$ and $B \in \mathscr{G}(X)$. If $A$ and $B$ are graded Morita equivalent, then $A$ and $B$ are isomorphic as graded $C^*$-algebras.*

PROOF. Suppose that $M$ is an $A$-$B$-equivalence bimodule. It will be shown that $w[\xi_A] = w[\xi_B]$. Let $\mathscr{U} = \{\mathscr{U}_i\}_{i\in I}$ be a locally finite open cover of $X$ with elements $a_i \in A^{(0)}$ chosen for each $i$, such that $x(a_i)$ is a degree 0 rank one projection for every $x \in \mathscr{U}_i$, and such that for each $i$, there exists $m_i \in A^{(0)}$ with $\langle m_i, m_i\rangle_A = a_i$. Property (b) of Definition 5.1 guarantees the existence of $m_i$.

Let $i, j \in I$. Suppose $x \in \mathscr{U}_i \cap \mathscr{U}_j$. Let $c_{ij} \in A$ be chosen as before. When $A$ and $B$ are strong Morita equivalent, there is a homeomorphism between $\widehat{A}$ and $\widehat{B}$ [**22**, 6.2.7]. Let $\hat{x}$ be an irreducible representation of $B$ associated to $x$ under this homeomorphism. Then $\hat{x}(\langle m_i, m_i\rangle_B)$ is a rank one projection for every $i$ [**11**]. Define $\hat{c}_{ij} = \langle m_i, c_{ij}m_j\rangle_B$. It is easy to check that $\hat{x}(\langle m_i, m_i\rangle_B\hat{c}_{ij}\langle m_j, m_j\rangle_B) = \hat{x}(\hat{c}_{ij}) \neq 0$. So $\hat{x}(\hat{c}_{ij})$ is a rank one operator with initial space equal to $\mathrm{Im}(\hat{x}\langle m_j, m_j\rangle_B)$ and range equal to $\mathrm{Im}(\hat{x}\langle m_i, m_i\rangle_B)$. Using the properties of Definition 5.1, one can compute that the $c_{ij}$ and the $\hat{c}_{ij}$ define the same cocycle in $C^2(\mathscr{U}; \underline{S}^1)$. Therefore, $w_2'(A) = w_2'(B)$ so $w_2^*[\xi_A] = w_2^*[\xi_B]$.

Since the $m_i$ and $m_j$ are chosen to be of degree 0, we can see that $\deg(\hat{c}_{ji}) = \deg(c_{ji})$. And $[A] = [B]$ in $\mathrm{GBr}^\infty(X)$ implies that $A$ and $B$ are isomorphic as graded $C^*$-algebras. □

5.3. COROLLARY. *Let $A$ and $B$ be separable, graded continuous trace $C^*$-algebras with spectrum $X$. Suppose that $A$ and $B$ are graded Morita equivalent. Then $A\hat{\otimes}\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ and $B\hat{\otimes}\mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ are isomorphic as graded $C^*$-algebras.*

**6. The proof of Proposition 2.4.** In the ungraded case, the fact that the continuous field $\xi_A$ constructed from a separable, stable, continuous trace $C^*$-algebra $A$ is a fiber bundle is based on [**3**, 10.7.11]. Proposition 2.4 is a graded version of this lemma. The proof of this proposition requires that we verify that the constructions in [**3**, Chapter 10, §§6–7] can be done in the graded setting.

Let $F_1$ be the category whose objects are pairs $(\mathscr{H}, e_0)$, where $\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$ is a graded Hilbert space and $e_0 \in \mathscr{H}^{(0)}$ is a unit vector. A morphism between $(\mathscr{H}, e_0)$ and $(\mathscr{H}', e_0')$ is a graded isomorphism $u\colon \mathscr{H} \to \mathscr{H}'$ such that $u(e_0) = e_0'$. Let $F_2$ be the category whose object are pairs $(A, p)$, where $A$ is a graded elementary $C^*$-algebra of infinite dimension and $p$ is a degree 0 projection of rank one. A morphism between $(A, p)$ and $(A', p')$ in $F_2$ is a graded isomorphism $g\colon A \to A'$ such that $g(p) = p'$. Note that, given a degree 0 projection of rank one on the graded Hilbert space $\mathscr{H}$, we may assume that it is a degree 0 projection whose image is in $\mathscr{H}^{(0)}$. Let the functor $\alpha\colon F_1 \to F_2$ be defined by $\alpha(\mathscr{H}, e_0) = (A, p)$, where $A = \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ and $p\colon \mathscr{H} \to \mathbf{C}e_0$ is the projection. If $u\colon (\mathscr{H}, e_0) \to (\mathscr{H}', e_0')$ is a morphism in $F_1$, then define $\alpha(u)\colon \mathscr{K}_{\mathrm{gr}}(\mathscr{H}) \to \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ by $\alpha(u)(T)(y) = u(T(u^{-1}(y)))$, for $T \in \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ and $y \in \mathscr{H}'$. It is easy to check that $\alpha(u)$ is a morphism in $F_2$.

Let $(\mathscr{H}, e_0) \in F_1$, and let $(A, p) = \alpha(\mathscr{H}, e_0)$. Then $Ap$ can be given an inner product by $(a, b)_{Ap} = (ae_0, be_0)_{\mathscr{H}}$. A grading on $Ap$ is defined as follows. Suppose $p_0\colon \mathscr{H}^{(0)} \to \mathbf{C}e_0$ is the projection. Let $\varsigma\colon \mathscr{H}^{(0)} \to \mathscr{H}^{(0)}$ and $\gamma\colon \mathscr{H}^{(0)} \to \mathscr{H}^{(1)}$ be maps. Then a typical element of $(Ap)^{(0)}$ has the form $\left(\begin{smallmatrix} \varsigma p_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and a typical element of $(Ap)^{(1)}$ has the form $\left(\begin{smallmatrix} 0 & 0 \\ \varsigma p_0 & 0 \end{smallmatrix}\right)$. An easy computation verifies that $(Ap)^{(i)}(Ap)^{(j)} \subset (Ap)^{(i+j)}$, for $i, j = 0, 1$. Suppose that $(\mathscr{H}, e_0) \in F_1$ and $\alpha(\mathscr{H}, e_0) = (A, p)$. If we define $\varphi\colon Ap \to \mathscr{H}$ by $\varphi(a) = ae_0$, for $a \in Ap$, then we can check that $\varphi$ is a graded isometric isomorphism.

Let $(A, p) \in F_2$, and construct the graded Hilbert space $Ap$. Note that $p$ is a unit vector of $Ap$. Then define a functor $\beta\colon F_2 \to F_1$ by $\beta(A, p) = (Ap, p)$. If $g\colon (A, p) \to (A', p')$ is a morphism of $F_2$, then $\beta(g)\colon (Ap, p) \to (A'p', p')$ is defined by $\beta(g)(ap) = g(a)p'$, for $a \in A$. Suppose the pair $(A, p)$ is an object of $F_2$. One has $\alpha\beta(A, p) = (\mathscr{K}(Ap), p)$. The homomorphism $\psi\colon A \to \mathscr{K}(Ap)$ defined by $\psi(a)(x) = ax$ for each $a \in A$ and $x \in Ap$, is a graded isomorphism.

The functors $\alpha$ and $\beta$ will now be extended to the case of continuous fields. Let $\xi(\mathscr{H}_x)$ be a continuous field of graded Hilbert spaces over $X$. Suppose that $s \in \Gamma(\xi(\mathscr{H}_x))$ such that $\|s(x)\| = 1$ for every $x \in X$, and that $s(x) \in \mathscr{H}_x^{(0)}$ for $x \in X$. Then $s$ is called a degree 0 unit section for $\xi(\mathscr{H}_x)$. Let $\xi$ be a continuous field of graded elementary $C^*$-algebras over $X$. An element $r \in \Gamma(\xi)$ is called a degree 0 rank one section if $r(x)$ is a degree 0 rank one projection for every $x \in X$. Let $\mathscr{F}_1$ be the category whose objects are pairs $(\xi(\mathscr{H}_x), s)$ where $\xi(\mathscr{H}_x)$ is a continuous field of graded Hilbert spaces over $X$ and $s$ is a degree 0 unit section of $\xi(\mathscr{H}_x)$. A morphism $\varsigma\colon (\xi(\mathscr{H}_x), s) \to (\xi(\mathscr{H}_x'), s')$ is defined by $\varsigma = \bigcup_{x\in X} \varsigma_x$, where $\varsigma_x\colon \mathscr{H}_x \to \mathscr{H}_x'$ is a graded isomorphism for every $x \in X$, and $\varsigma(s) = s'$. Let $\mathscr{F}_2$ be the category whose objects are pairs $(\xi, p)$ where $\xi$ is a continuous field of graded elementary $C^*$-algebras and where $p$ is a degree 0 rank one section for $\xi$. A morphism $\eta\colon (\xi, p) \to (\xi', p')$ is defined by $\eta = \bigcup_{x\in X} \eta_x$, where $\eta_x\colon \xi(x) \to \xi'(x)$ is a graded isomorphism for every $x \in X$ and $\eta(p) = p'$.

Suppose that $(\xi(\mathscr{H}_x), s) \in \mathscr{F}_1$. Then a degree 0 rank one section for the continuous field $\xi(\mathscr{K}_{\mathrm{gr}}(\mathscr{H}_x))$ can be constructed as follows. Let $r_s\colon X \to E(\xi(\mathscr{K}_{\mathrm{gr}}(\mathscr{H}_x)))$ by $r_s(x)(h) = (h, s(x))_{\mathscr{H}_x} s(x)$ where $h \in \mathscr{H}_x$. Then $r_s(x)$ is a degree 0 rank one projection for every $x \in X$. There is a functor $\alpha\colon \mathscr{F}_1 \to \mathscr{F}_2$ defined by $\alpha(\xi(\mathscr{H}_x), s) = (\xi(\mathscr{K}_{\mathrm{gr}}(\mathscr{H}_x)), r_s)$. If $\varsigma\colon (\xi(\mathscr{H}_x), s) \to (\xi(\mathscr{H}_x'), s')$ is a morphism of $\mathscr{F}_1$, then let $\alpha(\varsigma) = \bigcup_{x\in X} \alpha(\varsigma_x)$. The next result follows immediately.

6.1. LEMMA. *If $\varsigma\colon \xi(\mathscr{H}_x) \to \xi(\mathscr{H}_x')$ is a graded isomorphism, then the induced map $\alpha(\varsigma)\colon \xi(\mathscr{K}_{\mathrm{gr}}(\mathscr{H}_x)) \to \xi(\mathscr{K}_{\mathrm{gr}}(\mathscr{H}_x'))$ is a graded isomorphism.*

Let $(\xi, p) \in \mathscr{F}_2$ where $\xi(x) = A_x$. Define a functor $\beta\colon \mathscr{F}_2 \to \mathscr{F}_1$ by $\beta(\xi, p) = (\xi(A_x p(x)), p)$, where $p(x)$ is the unit vector of $A_x p(x)$ for every $x \in X$. If $\eta\colon (\xi, p) \to (\xi', p')$ is a morphism of $\mathscr{F}_2$, then let $\beta(\eta)$ be defined as $\beta(\eta) = \bigcup_{x\in X} \beta(\eta_x)$. The following lemma is a graded version of Definition 1.1.

6.2. LEMMA. *Let $A \in \mathscr{G}(X)$. For each $x \in X$, there exists an element $a \in A^{(0)}$ and a neighborhood $V_x$ of $x$ in $X$ such that, for every $v \in V_x$, $v(a)$ is a rank one projection of degree* 0.

PROOF. By Lemma 2.1, we may assume that the elements of $X$ are graded representations. Let $x \in X$ such that $x: A \to \mathscr{L}(\mathscr{H}_x)$, where $\mathscr{H}_x$ is a separable, graded, infinite-dimensional Hilbert space. Let $e_0 \in \mathscr{H}_x^{(0)}$ be a unit vector. Let $P_x$ be the degree 0 projection $\left(\begin{smallmatrix} p_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $p_0: \mathscr{H}_x^{(0)} \to \mathbf{C}e_0$. Since $\operatorname{Im}(x) = \mathscr{K}(\mathscr{H}_x)$, there exists $a_1 \in A$ with $x(a_1) = P_x$. We may assume that $\deg(a_1) = 0$ since $x$ is a graded homomorphism. Applying the proof of [**3**, 4.4.2] to $a_1$, we can construct an $a$ such that $x(a) = p_x$ and with the property that there exists a neighborhood $V_x$ of $x$ in $X$ such that $v(a)$ is a rank one projection for every $v \in V_x$. Then $\deg(a) = 0$ so $\deg v(a) = 0$ for every $v \in V_x$. □

6.3. LEMMA. *Let $\xi_A$ be the continuous field constructed from $A \in \mathscr{G}(X)$ as defined in* §2. *Then there exists an open cover $\{\mathscr{U}_i\}_{i \in I}$ of $X$ such that for every $i \in I$, there is a fiber-preserving, graded isomorphism*

$$h_i: \mathscr{U}_i \times \mathscr{K}_{\mathrm{gr}}(\mathscr{H}) \to \xi|_{\mathscr{U}_i}$$

*where $\mathscr{H}$ is a graded Hilbert space.*

PROOF. The continuous field $\xi_A$ has the following property: for each $x \in X$, there exists a neighborhood $V$ of $x$ and a map $p: V \to E(\xi_A)$ such that $p(y)$ is a degree 0 rank one projection for every $y \in V$ [**3**, 10.5.8]. Let $\{\mathscr{U}_i\}_{i \in I}$ be a locally finite open cover $X$ of such neighborhoods, with associated degree 0 rank one sections $p_i$. Let $\xi_A(x) = A_x$. The $\alpha$ and $\beta$ constructions for continuous fields imply that

$$\alpha\beta(\xi_A|_{\mathscr{U}_i}, p_i) = (\xi(\mathscr{K}_{\mathrm{gr}}(A_x p_i(x))), p_i).$$

Let $\psi_x: \xi_A|_{\{x\}} \to \mathscr{K}_{\mathrm{gr}}(A_x p_i(x))$ be the graded isomorphism constructed earlier for each $x \in X$. Let $\psi_i = \bigcup_{x \in \mathscr{U}_i} \psi_x$. By [**3**, 10.7.6(ii)], $\psi_i$ is an isomorphism. Then $k_i = \psi_i^{-1}$ is a fiber-preserving, graded isomorphism.

The algebra $A$ is stable, so $\xi_A$ is locally trivial of rank $\aleph_0$ [**21**, 1.12]. Then there is a graded isomorphism $\varphi_x: A_x p_i(x) \to \mathscr{H}$, where $\mathscr{H}$ is a separable, graded, infinite-dimensional Hilbert space. Let $\varphi_i = \bigcup_{x \in \mathscr{U}_i} \varphi_x$; by [**3**, 10.7.6(i)] and [**3**, 10.8.7], $\varphi_i$ is a graded isomorphism between trivial continuous fields of Hilbert spaces. Let $\varsigma_i = \alpha(\varphi_i^{-1})$; $\varsigma_i$ is graded by Lemma 6.1 and is clearly fiber-preserving. The coordinate function $h_i$ for $\xi_A$ can then be defined as:

$$h_i: \mathscr{U}_i \times \mathscr{K}_{\mathrm{gr}}(\mathscr{H}) \xrightarrow{\varsigma_i} \xi(\mathscr{K}_{\mathrm{gr}}(A_x p_i(x))) \xrightarrow{k_i} \xi_A|_{\mathscr{U}_i}. \quad \square$$

Every $h_{i,x}$ is a homeomorphism since it is a $^*$-isomorphism. An easy argument using the product topology for $\mathscr{U}_i \times \mathscr{K}_{\mathrm{gr}}(\mathscr{H})$ verifies that $h_i$ is a homeomorphism. Since each $h_i$ is graded, the composite $h_{i,x}^{-1} \circ h_{j,x}$ coincides with an element of $\operatorname{Aut}^0(\mathscr{K}) \approx \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$, for every $i$ and $j$. It is straightforward to verify that $g_{ij}: \mathscr{U}_i \cap \mathscr{U}_j \to \mathscr{P}\mathscr{U}_{\mathrm{gr}}(\mathscr{H})$ is continuous.

This completes the proof of Proposition 2.4. □

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DEPARTMENT OF MATHEMATICS, WELLESLEY COLLEGE, WELLESLEY, MASSACHUSETTS 02181

*Current address*: Department of Mathematics and Computer Science, DePauw University, Greencastle, Indiana 46135